38406501359372282063949 & all that: Monodromy of Fano Problems

Sachi Hashimoto
joint with Borys Kadets

Boston University

ICERM
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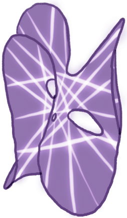
Set-up

A *Fano problem* $([d], n, r)$ is the problem of enumerating all r-dimensional linear subspaces on a general complete intersection $X_{[d]} = X_{d_1} \cap \cdots \cap X_{d_s} \subset \mathbb{P}^n_K$ whenever this is finite.



Example

- $N((3), 3, 1) = 27$
- $N((3), 18, 6) = 384065013593722820639 49$ (H.-Kadets)

Monodromy group

Fix a Fano problem $([d], n, r)$.

- $M_{[d]} := \prod_{i=1}^{s} \mathbb{P}^{\binom{d_i+n}{n}-1}$ moduli space of tuples of hypersurfaces of degrees $[d]$.
- Incidence scheme $I := \{(X_{[d]}, \Lambda) | \Lambda \subset X_{[d]}\} \subset M_{[d]} \times \mathbb{G}(r, n)$
- $\pi : I \to M_{[d]}$ is generically étale: let $G_{([d],n,r)}$ be the *Fano monodromy group*.

Motivational question

Question

What can we say about the Fano monodromy group?

Basic principle: we need geometry to compute the monodromy group, which in turn reveals information about the geometry.

Previous work

Example (Harris, 1979)

The Fano monodromy group of the cubic surface is the Weyl group $G_{((3),3,1)} = W(E_6) \subset S_{27}$.
The Fano monodromy groups of lines in hypersurfaces $G_{((d),n,1)}$ are symmetric groups $S_{N((d),n,1)}$ when $(d) \neq (3)$.

Main Theorems

Theorem (H.-Kadets)

Suppose we are not in the case of a cubic surface in $\mathbb{P}^3$ or the intersection of two quadrics in $\mathbb{P}^n$. Then the Fano monodromy group is the alternating group or the symmetric group, $A_{N([d],n,r)}$ or $S_{N([d],n,r)}$.

Theorem (H.-Kadets)

Suppose $\operatorname{char} K \neq 2$*. The Fano monodromy group of k-planes on the complete intersection of two quadrics in $\mathbb{P}^{2k+2}$ is the Weyl group $W(D_{2k+3})$.*

Group theory

There is a good classification of multiply transitive groups: if G is a 4-transitive permutation group which acts on a set of size $n > 24$, it must be A_n or S_n.

A group G acting on a set S is $(m+1)$-transitive if the stabilizer of a set of m points $T \subset S$ acts transitively on $S \setminus T$.

Transitivity

An étale covering $f : X \to Y$ of a normal irreducible scheme Y has transitive monodromy iff X is irreducible.

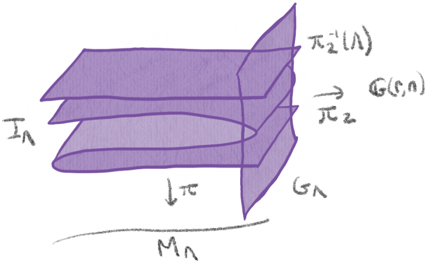
Our situation:

$$\begin{tikzcd} I \arrow[r, "\pi_2"] \arrow[d, "\pi"'] & \mathbb{G}(r,n) \\ M_{[d]} & \end{tikzcd}$$

Double Transitivity

Fix an r-plane Λ.
Consider M_Λ, parametrizing $X_{[d]}$ containing Λ.

$\pi : I_\Lambda \to M_\Lambda$ has monodromy group G_Λ contained in a 1-point stabilizer. Show it acts transitively on a smooth fiber of $I' := I_\Lambda \setminus \pi_2^{-1}(\Lambda)$.

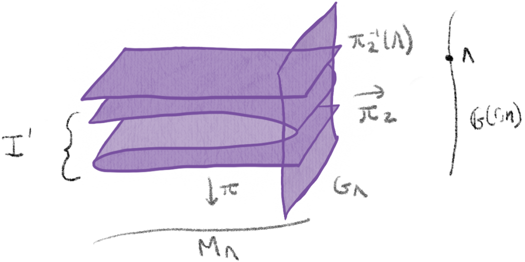


Equidimensionality

Optimistic approach: $\pi_2 : I' \to \mathbb{G}(r,n)$, hope this is proper with irreducible *equidimensional* fibers over an open set of $\mathbb{G}(r,n)$.

Modify I' to make this true: $U := \{\Sigma | \Sigma \cap \Lambda = \emptyset\} \subset \mathbb{G}(r,n)$, and $I'' := I' \cap \pi_2^{-1}(U)$ has equidimensional fibers over U.

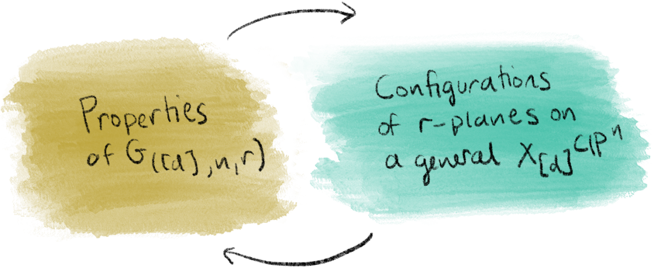
For a general $X_{[d]}$, a dimension count shows $\pi^{-1}(X_{[d]})$ is contained in $\pi_2^{-1}(U)$, so suffices to show I'' irreducible.



From monodromy to geometry

Remark

Double transitivity of the Fano monodromy group $G_{([d],n,r)}$ implies that any two distinct r-planes Λ_1 and Λ_2 on a general complete intersection $X_{[d]}$ are linearly independent.



Inductive approach

As long as there is room for m linearly independent r-planes, we can go from m-transitive to $(m+1)$-transitive by considering $\pi : I_\Lambda \to M_\Lambda$ fixing $\Lambda_1, \dots, \Lambda_m$ contained in $X_{[d]}$. Monodromy group G_Λ contained in m-point stabilizer.

double transitivity $\to$

any Λ_1, Λ_2 on a general $X_{[d]}$ are linearly independent

$\downarrow$

$\Lambda_1, \Lambda_2, \Lambda$ pairwise linearly independent
$\Rightarrow$ dim $H^0(\mathcal{I}_{\Lambda_1 \cup \Lambda_2 \cup \Lambda}(d))$ is constant

$\swarrow$

triple transitivity

$\to$ ...

Small Fano problems

Sometimes the induction stops short: 6-planes on $X_{(3)} \subset \mathbb{P}^{18}$, $18+1 < 3(6+1)$, can't fit 3 linearly independent planes to prove 4-transitivity.

Group theory fact: the only 3-transitive groups that act on sets with cardinality $n > 24$ and n not a power 2 or one more than a power of a prime are A_n and S_n.
So we must compute $N((3), 18, 6) =? = \deg(F_6(X_{(3)}))$.

There is a finite list of "small Fano *problems*", $[d] \neq (2,2,2)$.

Small Fano problems

By Debarre-Manivel, $N((3), 18, 6)$ is the coefficient of $x_0^{18} x_1^{17} \dots x_6^{12}$ in

$$\left(\prod_{a_0 + a_1 + \cdots + a_6 = 3} a_0 x_0 + \cdots + a_6 x_6 \right) \prod_{0 \le i < j \le 6} (x_i - x_j).$$

Has 105 linear factors, after multiplying only 50 together, have roughly 1 million monomials...

$N((3), 22, 7)$ is an *even bigger* small problem!